

QUASI-MORPHISMS ON THE GROUP OF AREA-PRESERVING DIFFEOMORPHISMS OF THE 2-DISK VIA BRAID GROUPS

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ABSTRACT. Recently Gambaudo and Ghys proved that there exist infinitely many quasi-morphisms on the group $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ of area-preserving diffeomorphisms of the 2-disk D^2 . For the proof, they constructed a homomorphism from the space of quasi-morphisms on the braid group to the space of quasi-morphisms on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$. In this paper, we study this homomorphism and prove its injectivity.

1. INTRODUCTION

For a group G , a function $\phi: G \rightarrow \mathbb{R}$ is called a *quasi-morphism* if the real valued function on $G \times G$ defined by

$$(g, h) \mapsto \phi(gh) - \phi(g) - \phi(h)$$

is bounded. The real number

$$D(\phi) = \sup_{g, h \in G} |\phi(gh) - \phi(g) - \phi(h)|$$

is called the *defect* of ϕ . We denote the \mathbb{R} -vector space of quasi-morphisms on the group G by $\hat{Q}(G)$. By definition, bounded functions on groups are quasi-morphisms. Hence we denote the set of bounded functions on the group G by $C_b^1(G; \mathbb{R})$ and consider the quotient space $Q(G) = \hat{Q}(G)/C_b^1(G; \mathbb{R})$. A quasi-morphism $\phi: G \rightarrow \mathbb{R}$ is said to be *homogeneous* if the equation

$$\phi(g^p) = p \phi(g)$$

holds for any $g \in G$ and $p \in \mathbb{Z}$. For any quasi-morphism ϕ , a homogeneous quasi-morphism $\tilde{\phi}$ is defined by setting

$$\tilde{\phi}(g) = \lim_{p \rightarrow \infty} \frac{1}{p} \phi(g^p).$$

The limit always exists for each element g of G . The new function $\tilde{\phi}$ is in fact a quasi-morphism equal to the original quasi-morphism ϕ as an element of $Q(G)$. Thus we can identify the vector space of homogeneous quasi-morphisms on the group G with $Q(G)$. Homogeneous quasi-morphisms are invariant under conjugations. Therefore we are interested in $Q(G)$ rather than $\hat{Q}(G)$.

Let $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ be the group of area-preserving C^∞ -diffeomorphisms of the 2-disk D^2 , which are the identity on a neighborhood of the boundary. On the vector space $Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$, the following theorem is known.

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Theorem 1.1 (Entov-Polterovich [2], Gambaudo-Ghys [4]). *The vector space $Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is infinite dimensional.*

To prove Theorem 1.1, Entov and Polterovich explicitly constructed uncountably many quasi-morphisms on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$, which are linearly independent. After that Gambaudo and Ghys constructed countably many quasi-morphisms on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ by a different idea, which is to consider the suspension of area-preserving diffeomorphisms of the disk and average the value of the signature of the braids appearing in the suspension. By generalizing their strategy Brandenbursky [1] defined the homomorphism

$$\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2)),$$

which we review in Section 2. Here, $P_n(D^2)$ denotes the pure braid group on n -strands.

Let $B_n(D^2)$ be the braid group on n -strands. The natural inclusion $i: P_n(D^2) \rightarrow B_n(D^2)$ induces the homomorphism $Q(i): Q(B_n(D^2)) \rightarrow Q(P_n(D^2))$. In this paper, we study the homomorphism Γ_n and prove the following theorem.

Theorem 1.2. *The composition*

$$\Gamma_n \circ Q(i): Q(B_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$$

is injective.

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2. GAMBAUDO AND GHYS' CONSTRUCTION AND PROOF OF THE MAIN THEOREM

In this section, we review Gambaudo and Ghys' construction [4] of quasi-morphisms on the group $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ in a generalized form and prove Theorem 1.2.

Let $X_n(D^2)$ be the configuration space of ordered n -tuples in the 2-disk D^2 and $x^0 = (x_1^0, \dots, x_n^0)$ its base point. For any $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$ and for almost alls $x = (x_1, \dots, x_n) \in X_n(D^2)$, we define the pure braid $\gamma(g; x)$ as the following. First we set the loop $l(g; x): [0, 1] \rightarrow X_n(D^2)$ by

$$l(g; x)(t) = \begin{cases} \{(1-3t)x_i^0 + 3tx_i\} & (0 \leq t \leq \frac{1}{3}) \\ \{g_{3t-1}(x_i)\} & (\frac{1}{3} \leq t \leq \frac{2}{3}) \\ \{(3-3t)g(x_i) + (3t-2)x_i^0\} & (\frac{2}{3} \leq t \leq 1) \end{cases},$$

where $\{g_t\}_{t \in [0,1]}$ is a Hamiltonian isotopy such that g_0 is the identity and $g_1 = g$. We define the pure braid $\gamma(g; x)$ to be the braid represented by the loop $l(g; x)$. For almost every x , the braid $\gamma(g; x)$ is well-defined. Furthermore, the braid $\gamma(g; x)$ is independent of the choice of the flow $\{g_t\}$. This is because of the fact

the group $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ is contractible, which is easily proved from the contractibility of the diffeomorphism group $\text{Diff}^\infty(D^2, \partial D^2)$ of D^2 [7] and the homotopy equivalence between $\text{Diff}^\infty(D^2, \partial D^2)$ and $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ [6]. For a quasi-morphism ϕ on the pure braid group $P_n(D^2)$ on n -strands, we define the function $\hat{\Gamma}_n(\phi): \text{Diff}_\Omega^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ by

$$\hat{\Gamma}_n(\phi)(g) = \int_{x \in X_n(D^2)} \phi(\gamma(g; x)) dx.$$

For any $\phi \in Q(P_n(D^2))$ and $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$ the function $\phi(\gamma(g; \cdot))$ is integrable and thus the map $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is well-defined [1]. The obtained function $\hat{\Gamma}_n(\phi): \text{Diff}_\Omega^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ is also a quasi-morphism and the map $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is clearly \mathbb{R} -linear. Moreover, it is easily checked that any bounded function on $P_n(D^2)$ is mapped to a bounded function on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ and thus the homomorphism $\hat{\Gamma}: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ induces the homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$.

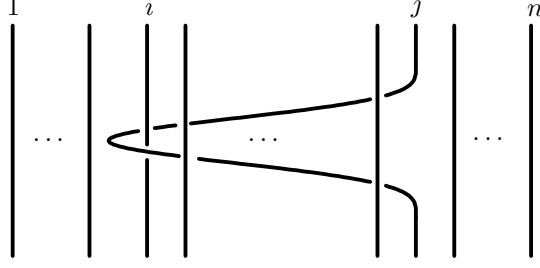
Remark 2.1. It is easy to see that the homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ maps the classical linking number homomorphism $\text{lk}_n: B_n(D^2) \rightarrow \mathbb{R}$ on the braid group to a homomorphism on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$. In fact, the image of $\text{lk}: B_n(D^2) \rightarrow \mathbb{R}$ by the homomorphism $\Gamma_n(\text{lk}_n)$ coincides with a constant multiple of the classical Calabi homomorphism on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ [3] and in this sense quasi-morphisms obtained in this way can be considered as generalizations of the Calabi homomorphism. By an argument of Brandenbursky, which verify that the homomorphism $\Gamma: \hat{Q}(P_n) \rightarrow \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is well-defined, it is observed that quasi-morphisms obtained by the homomorphism $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ can be defined on the group of area-preserving C^1 -diffeomorphisms of D^2 , as well as the Calabi homomorphism.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let us suppose that a homogeneous quasi-morphism $\phi \in \hat{Q}(B_n(D^2))$ is non-trivial. Then there exists a braid $\beta \in B_n(D^2)$ such that $\phi(\beta) \neq 0$. We may assume that β is pure. It is sufficient to prove that the homogeneous quasi-morphism $\hat{\Gamma}_n(\phi) \in \hat{Q}(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is also non-trivial. That is, there exists an area-preserving diffeomorphism $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$ such that

$$\lim_{p \rightarrow \infty} \frac{1}{p} \Gamma_n(\phi)(g^p) \neq 0.$$

Let $A_{i,j}$ be the pure braid which twists only the i -th and the j -th strands for $1 \leq i < j \leq n$ (see Figure 1). Since the braid β is pure, it can be written as a composition of $A_{i,j}$'s and their inverses. We take n disjoint subsets U_i 's of D^2 . Furthermore, for a pair of (i, j) , we take subsets $V_{i,j}$ and $W_{i,j}$ of D^2 such that $U_i \cup U_j \subset W_{i,j} \subset V_{i,j}$, $U_k \cap V_{i,j} = \emptyset$ if $k \neq i, j$ and $V_{i,j}, W_{i,j}$ are diffeomorphic to D^2 . Let $\{h_t\}_{t \in [0,1]}$ be a path in $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ such that the support of h_t is contained in the interior of $V_{i,j}$ and rotates $W_{i,j}$ once. Taking paths $\{h_t\}$'s constructed above for the all $A_{i,j}$'s which present β and composing them, we have a path $\{g_t\}_{t \in [0,1]}$ in $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ with $g_0 = \text{id}$ which twists U_i 's in the form of the pure braid β . If we set $g = g_1$, then g is the identity on U_i 's and $\gamma(g; (x_1, \dots, x_n)) = \beta$ for $x_i \in U_i$.

FIGURE 1. pure braid $A_{i,j}$

Then by setting $U = U_1 \cup \dots \cup U_n$, we have

$$\begin{aligned}
 & \lim_{p \rightarrow \infty} \frac{1}{p} \hat{\Gamma}_n(\phi)(g^p) \\
 &= \lim_{p \rightarrow \infty} \frac{1}{p} \left(\int_{x \in X_n(U)} \phi(\gamma(g^p; x)) dx + \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x)) dx \right) \\
 &= \int_{x \in X_n(U)} \phi(\gamma(g; x)) dx + \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x)) dx.
 \end{aligned}$$

If we denote the first term of the equation by Y and set $a_i = \text{area}(U_i)$ and $[n] = \{1, \dots, n\}$, then Y is written as

$$\int_{x \in X_n(U)} \phi(\gamma(g; x)) dx = \sum_{F: [n] \rightarrow [n]} \left(\prod_{i=1}^n a_{F(i)} \right) x_F,$$

where $x_F = \phi(\gamma_F)$ and $\gamma_F = \gamma(g; x)$ for x in the case when each x_i is in $U_{F(i)}$. The real numbers x_F 's have the following properties.

- (i) For two maps F and $G: [n] \rightarrow [n]$, if $\#F^{-1}(i) = \#G^{-1}(i)$ for each $1 \leq i \leq n$ then $x_F = x_G$.
- (ii) If a map $F: [n] \rightarrow [n]$ is bijective, then x_F is non-zero.

The property (i) follows from the invariance of ϕ under conjugation and the property (ii) follows because $\phi(\beta)$ is non-zero. Therefore, the coefficient of $a_1 \dots a_n$ in Y is non-zero. Since the polynomial Y is not identically 0, we can choose a_i 's s that Y is non-zero.

Note that if we replace a_i 's by bigger ones fixing the ratio of any two of them the term Y stays non-zero. On the other hand, the values $\phi(\gamma(g; x))$ is bounded because of the construction of g , and we thus have

$$\lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \notin X_n(U)} \phi(\gamma(g^p; x)) dx \rightarrow 0 \quad (\text{as } a_1 + \dots + a_n \rightarrow \text{area}(D^2)).$$

This completes the proof. \square

As we noted in Remark 2.1, The homomorphism $\hat{\Gamma}_n$ maps any homomorphism on $P_n(D^2)$ to a homomorphism on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$. Hence the homomorphism

$$Q(P_n(D^2))/H^1(P_n(D^2); \mathbb{R}) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))/H^1(\text{Diff}_\Omega^\infty(D^2, \partial D^2); \mathbb{R})$$

is also induced. By an argument similar to the proof of Theorem 1.2, the following proposition holds.

Proposition 2.2. *The map*

$$Q(B_n(D^2))/H^1(B_n(D^2); \mathbb{R}) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))/H^1(\text{Diff}_\Omega^\infty(D^2, \partial D^2); \mathbb{R})$$

induced by the composition $\Gamma_n \circ Q(i): Q(B_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is injective.

The homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ can be defined also for the 2-sphere S^2 instead of D^2 as Gambaudo and Ghys mentioned in their paper. Let $\text{Diff}_\Omega^\infty(S^2)_0$ be the identity component of the group of area-preserving diffeomorphisms of S^2 . Then we can choose a pure braid $\gamma(g; x) \in P_n(S^2)$ for any $g \in \text{Diff}_\Omega^\infty(S^2)_0$ and for almost every $x \in X_n(S^2)$ as in the case of the 2-disk. Since the group $\text{Diff}_\Omega^\infty(S^2)_0$ is homotopy equivalent to $SO(3)$ [6][7] and its fundamental group has order 2, for any element g of $\text{Diff}_\Omega^\infty(S^2)_0$ there exist two homotopy classes of paths connecting the identity and g in $\text{Diff}_\Omega^\infty(S^2)_0$. However, for any homogeneous quasi-morphism ϕ on $P_n(S^2)$, the value $\phi(\gamma(g; x))$ is independent of the choice of the path. In fact, from a path which represents the generator of $\pi_1(\text{Diff}_\Omega^\infty(S^2)_0)$ has order 2 and is in the center of $P_n(S^2)$. Hence the homomorphism $\Gamma_n: Q(P_n(S^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(S^2)_0)$ is defined. Since the braid group $B_n(S^2)$ of the 2-sphere on n -strands can be considered as a quotient group of the braid group $B_n(D^2)$, by an argument similar to the proof of Theorem 1.2, we obtain the following theorem.

Theorem 2.3. *The composition*

$$\Gamma_n \circ Q(i): Q(B_n(S^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(S^2)_0)$$

is injective.

The homomorphism $Q(i)$ in the statement of Theorem 2.3 is the one induced from the inclusion $i: P_n(S^2) \rightarrow B_n(S^2)$.

3. KERNEL OF THE HOMOMORPHISM Γ_n

The homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ itself is not injective although Theorem 1.2 holds. In this section we study the kernel of the homomorphism Γ_n .

Let G be a group and H its finite index subgroup. We denote by $\overline{\beta}$ the image of an element $\beta \in G$ by the natural projection $G \rightarrow G/H$. For each left coset $\sigma \in G/H$ of G modulo H , we fix an element $\gamma_\sigma \in G$ such that $\overline{\gamma_\sigma} = \sigma$ and for any $\phi \in \hat{Q}(H)$ define the function $\hat{T}(\phi): G \rightarrow \mathbb{R}$ by

$$\hat{T}(\phi)(\beta) = \frac{1}{(G:H)} \sum_{\sigma \in G/H} \phi(\gamma_{\overline{\beta\gamma_\sigma}}^{-1} \beta \gamma_\sigma).$$

Since $\gamma_{\overline{\beta\gamma_\sigma}}^{-1} \beta \gamma_\sigma$ is in H , the function $\hat{T}(\phi)$ is well-defined on G .

Lemma 3.1. *For any quasi-morphism ϕ on H , the function $\hat{T}(\phi): G \rightarrow \mathbb{R}$ is also a quasi-morphism.*

Proof. Since the equality

$$\gamma_{\beta_1\beta_2\gamma_\sigma}^{-1}\beta_1\beta_2\gamma_\sigma = (\gamma_{\beta_1\beta_2\gamma_\sigma}^{-1}\beta_1\gamma_{\beta_2\gamma_\sigma})(\gamma_{\beta_2\gamma_\sigma}^{-1}\beta_2\gamma_\sigma)$$

holds, we have the inequality

$$\begin{aligned} & |\hat{\mathcal{T}}(\phi)(\beta_1\beta_2) - \hat{\mathcal{T}}(\phi)(\beta_1) - \hat{\mathcal{T}}(\phi)(\beta_2)| \\ &= \frac{1}{(G:H)} \left| \sum_{\sigma \in G/H} \left\{ \phi((\gamma_{\beta_1\beta_2\gamma_\sigma}^{-1}\beta_1\gamma_{\beta_2\gamma_\sigma})(\gamma_{\beta_2\gamma_\sigma}^{-1}\beta_2\gamma_\sigma)) \right. \right. \\ &\quad \left. \left. - \phi(\gamma_{\beta_1\gamma_\sigma}^{-1}\beta_1\gamma_\sigma) - \phi(\gamma_{\beta_2\gamma_\sigma}^{-1}\beta_2\gamma_\sigma) \right\} \right| \\ &= \frac{1}{(G:H)} \left| \sum_{\sigma \in G/H} \left\{ \phi((\gamma_{\beta_1\beta_2\gamma_\sigma}^{-1}\beta_1\gamma_{\beta_2\gamma_\sigma})(\gamma_{\beta_2\gamma_\sigma}^{-1}\beta_2\gamma_\sigma)) \right. \right. \\ &\quad \left. \left. - \phi(\gamma_{\beta_1\beta_2\gamma_\sigma}^{-1}\beta_1\gamma_{\beta_2\gamma_\sigma}) - \phi(\gamma_{\beta_2\gamma_\sigma}^{-1}\beta_2\gamma_\sigma) \right\} \right| \\ &\leq D(\phi). \end{aligned}$$

Hence the function $\hat{\mathcal{T}}(\phi): G \rightarrow \mathbb{R}$ is also a quasi-morphism. \square

The map $\hat{\mathcal{T}}: \hat{Q}(H) \rightarrow \hat{Q}(G)$ is clearly \mathbb{R} -linear and induces a homomorphism $\mathcal{T}: Q(P_n(D^2)) \rightarrow Q(B_n(D^2))$. Furthermore, the following proposition holds.

Proposition 3.2. *The homomorphism $\mathcal{T}: Q(H) \rightarrow Q(G)$ is independent of the choice of γ_σ 's.*

Proof. Suppose that ϕ is a homogeneous quasi-morphism on H . If an element β is in H , then $\gamma_\sigma\beta = \sigma$ for each $\sigma \in G/H$. For any $\beta \in G$ there exists an integer k such that β^k is in H and we have

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{1}{p} \hat{\mathcal{T}}(\phi)(\beta^p) &= \lim_{p' \rightarrow \infty} \frac{1}{kp'} \hat{\mathcal{T}}(\phi)(\beta^{kp'}) \\ &= \lim_{p' \rightarrow \infty} \frac{1}{(G:H)kp'} \sum_{\sigma \in G/H} \phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma)^{p'} \\ &= \frac{1}{(G:H)k} \sum_{\sigma \in G/H} \phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma). \end{aligned} \tag{3.1}$$

Since ϕ is invariant under conjugations in H , the value $\phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma)$ depends only on σ . \square

Let $Q(i): Q(G) \rightarrow Q(H)$ be the homomorphism induced by the inclusion $i: H \rightarrow G$. As a corollary to Equality (3.1), we have the following.

Corollary 3.3. *The composition $\mathcal{T} \circ Q(i): Q(G) \rightarrow Q(G)$ is the identity on $Q(G)$. Furthermore, we have the decomposition*

$$Q(H) = \text{Ker}(\mathcal{T}) \oplus \text{Im}(Q(i))$$

as vector spaces.

Remark 3.4. Of course, the homomorphism $\hat{\mathcal{T}}(\phi): G \rightarrow \mathbb{R}$ can be defined using the right coset $H \backslash G$ instead of G/H by

$$\hat{\mathcal{T}}(\phi)(\beta) = \frac{1}{(G:H)} \sum_{\sigma \in G/H} \phi(\gamma_\sigma \beta \gamma_{\gamma_\sigma \beta}^{-1}).$$

By an argument similar to the proof of Lemma 3.1 and Proposition 3.2, it is verified that this alternative definition is also well-defined and induces the same homomorphism $\mathcal{T}: Q(H) \rightarrow Q(G)$.

Remark 3.5. The homomorphism $\mathcal{T}: Q(H) \rightarrow Q(G)$ is just a straightforward generalization of transfer map, and it is also introduced in [5] and [8].

Since the pure braid groups $P_n(D^2)$ and $P_n(S^2)$ are finite index subgroups of the braid groups $B_n(D^2)$ and $B_n(S^2)$, respectively, the homomorphisms

$$\mathcal{T}: Q(P_n(D^2)) \rightarrow Q(B_n(D^2)) \quad \text{and} \quad \mathcal{T}: Q(P_n(S^2)) \rightarrow Q(B_n(S^2))$$

can be defined and Corollary 3.3 is true for $G = B_n(D^2), H = P_n(D^2)$ and $G = B_n(S^2), H = P_n(S^2)$, respectively.

The following proposition is the main result of this section.

Proposition 3.6. *The composition*

$$\Gamma_n \circ Q(i) \circ \mathcal{T}: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$$

coincides with Γ_n . In particular, $\text{Ker}(\Gamma_n) = \text{Ker}(\mathcal{T})$ and $\text{Im}(\Gamma_n) = \text{Im}(\Gamma_n \circ Q(i))$.

Proof. Let \mathfrak{S}_n be the symmetric group of n symbols. By Equality (3.1), for any homogeneous quasi-morphism $\phi \in Q(P_n(D^2))$ and any area-preserving diffeomorphism $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$,

$$\lim_{p \rightarrow \infty} \frac{1}{p} \hat{\Gamma}_n \circ Q(i) \circ \hat{\mathcal{T}}(\phi)(g^p) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; x) \gamma_\sigma^{-1}) dx. \quad (3.2)$$

For any $\sigma \in \mathfrak{S}_n$ and almost all $x \in D^2$, we set the path $l: [0, 1] \rightarrow X_n(D^2)$ by

$$l(t) = \begin{cases} \{(1-2t)x_i^0 + 2tx_i\} & (0 \leq t \leq \frac{1}{2}) \\ \{(2-2t)x_i + (2t-1)x_{\sigma(i)}^0\} & (\frac{1}{2} \leq t \leq 1) \end{cases}.$$

Considering the path l as a loop in the quotient space $X_n(D^2)/\mathfrak{S}_n$, we define the braid $\beta(\sigma; x)$ to be the braid represented by the loop l . Then by definition,

$$\beta(\sigma; x) \gamma(g; \sigma^{-1}(x)) \beta(\sigma; g_* x)^{-1} = \gamma(g; x),$$

where the symmetric group \mathfrak{S}_n acts on $X_n(D^2)$ by the permutation

$$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Since the homomorphism $\mathcal{T}: Q(P_n(D^2)) \rightarrow Q(B_n(D^2))$ is defined independently to the choice of braids γ_σ 's, we may choose γ_σ to be $\beta(\sigma; x)$. Hence we have

$$\begin{aligned} \gamma_\sigma \gamma(g; \sigma^{-1}(x)) \gamma_\sigma^{-1} &= \beta(\sigma; x) \gamma(g; \sigma^{-1}(x)) \beta(\sigma; x)^{-1} \\ &= \gamma(g; x) \beta(\sigma; g_*(x)) \beta(\sigma; x)^{-1}. \end{aligned}$$

Since the function $\phi(\beta(\sigma; \cdot)): D^2 \rightarrow \mathbb{R}$ is bounded on D^2 , we have

$$\begin{aligned} & \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; x) \gamma_\sigma^{-1}) dx \\ &= \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; \sigma^{-1}(x)) \gamma_\sigma^{-1}) dx \\ &= \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma(g^p; x)) dx. \end{aligned}$$

Therefore, by Equality (3.2),

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{1}{p} \hat{\Gamma}_n \circ Q(i) \circ \hat{T}(\phi)(g^p) &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \lim_{p \rightarrow \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma(g^p; x)) dx \\ &= \lim_{p \rightarrow \infty} \frac{1}{p} \hat{\Gamma}_n(\phi)(g^p) \end{aligned}$$

and thus we have $\Gamma_n \circ Q(i) \circ \mathcal{T} = \Gamma_n$.

Then obviously $\text{Ker}(\mathcal{T}) \subseteq \text{Ker}(\Gamma_n)$ and $\text{Im}(\Gamma_n) = \text{Im}(\Gamma_n \circ Q(i))$ hold. If $\phi \in \text{Ker}(\Gamma_n)$ then

$$\Gamma_n \circ Q(i) \circ \mathcal{T}(\phi) = \Gamma_n(\phi) = 0$$

and hence $\mathcal{T}(\phi) = 0$ by Theorem 1.2. Thus we have $\text{Ker}(\Gamma_n) \subseteq \text{Ker}(\mathcal{T})$. \square

Remark 3.7. Proposition 3.6 also holds for $P_n(S^2)$ and $\text{Diff}_\Omega^\infty(S^2)_0$ instead of $P_n(D^2)$ and $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$, respectively.

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